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## Self-duality in triplet Potts models

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**Abstract.** We obtain a class of self-dual  $q$ -state models generalizing the triangular lattice triplet Ising model in the same way that the standard  $q$ -state Potts model generalizes the self-dual square lattice Ising model. For four-state models we find an additional class of triplet models that are self-dual at a single temperature in analogy with the Ashkin–Teller model.

### 1. Introduction

It has become apparent that two-dimensional lattice models can exhibit extremely diverse critical behaviour. One of the best known of these systems is the eight-vertex model which was shown to have its critical exponents varying continuously with the interaction strengths (Baxter 1971). Jungling and Obermair (1974) have shown that the eight-vertex model can be represented by an Ising model with competing one-spin and two-spin interactions, so that the property of continuously varying exponents does not depend on the occurrence of multi-spin interactions. It appears that in two dimensions the critical exponents of the  $q$ -state Potts model (Potts 1952) vary continuously with  $q$ . (The random cluster formulation of Fortuin and Kasteleyn (1972) is used to define the Potts model for non-integral  $q$ .)

There is very little guidance as to how all these two-dimensional models can be classified, although the concept of weak universality classes having fixed values of exponent combinations such as  $\beta/\nu$ ,  $\delta$ ,  $(2-\alpha)/\nu$  etc (Suzuki 1974) is a possible beginning for a classification. In view of the apparent importance of the symmetries of a model in understanding its critical behaviour, models having particularly high symmetry may repay investigation. As an example, Enting (1975) has suggested that the four-state standard Potts model has such high symmetry that the critical exponents could be derived from general exponent relations.

In the present work we consider the special symmetry of models that are self-dual so that the high-temperature partition function can be obtained from the low-temperature partition function by a simple change of variable. The earliest example was the square lattice Ising model (Kramers and Wannier 1941) which was generalized to the square lattice  $q$ -state Potts model by Potts (1952). It is also possible to obtain a self-dual model on a triangular lattice by considering interactions proportional to the product of triplets of Ising spins (Merlini and Gruber 1972, Wood and Griffiths 1972).

We show here how the triplet model can be generalized to a self-dual  $q$ -state triplet model and discuss alternative ways of formulating the duality transformation. Section 2

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discusses the model and the direct graphical transformation. Section 3 extends these results to a triplet Ashkin–Teller model analogue. Section 4 formulates the  $q$ -state model in a random cluster formulation.

## 2. Graphical expansions and transformations

In most graphical expansions for lattice models, the graphs are built from elements that cover the sites involved in the corresponding interactions. For example, one-spin (field) interactions are represented by points and two-spin interactions are represented by lines. For high-temperature expansions the graphs typically represent the actual contributions to  $\exp(-\beta\mathcal{H})$ , while for low-temperature expansions the graph elements represent perturbations from some ground state.

The individual graph elements, the points, lines, etc, are combined to form graphs that are usually subject to some constraint (§ 4 considers ‘unconstrained’ expansions). A typical constraint is that of the zero-field spin  $\frac{1}{2}$  Ising model which has the requirement that all allowed graphs shall have an even number of lines meeting at each vertex. Wegner (1973) has shown how such local constraints can be used to generate transformations to a new set of local variables, and a new set of constraints describing the graphical expansion of the transformed system. For self-dual models we want a transformation to an equivalent set of local variables and an equivalent set of constraints.

For the  $q$ -state model with triplet interactions the basic graph elements are triangles labelled 0 to  $q-1$ . For both high- and low-temperature expansions we propose the constraint: the sum of the indices of the six triangles meeting at any vertex is an integral multiple of  $q$ . This rule will apply to a model in which each site  $j$  has a variable  $t_j$  that can be in one of  $q$  states 0 to  $q-1$ . The energy around each elementary triangle  $jkl$  is

$$E_{jkl} = \begin{cases} 0 & \text{if } t_j + t_k + t_l = 0 \pmod{q} \\ J & \text{otherwise.} \end{cases} \quad (1)$$

The graphical representation for the low-temperature expansion involves dividing the faces of the triangular lattice according to the parity of the triangle (ie separating ‘up’ pointing triangles from ‘down’ pointing triangles). For the graphical expansion, triangles of one parity have their graph indices given by  $(t_j + t_k + t_l) \pmod{q}$ , while triangles of the other parity have their graph indices given by  $(-t_j - t_k - t_l) \pmod{q}$ . The constraint above follows by simple addition.

The weights of graphs are given by the products of the weights of the basic elements, faces of index 0 contributing 1, and other faces contributing  $u = \exp(-\beta J)$ . The low-temperature (LT) expansion is

$$Z_{\text{LT}} = q^2 \sum_{\substack{\text{allowed} \\ \text{graphs}}} \prod_{\substack{\text{nonzero} \\ \text{faces}}} u. \quad (2)$$

The factor  $q^2$  corresponds to the number of ground states since any pair of neighbours can be set into arbitrary states, but this choice will then determine the ground state for the whole lattice.

The high-temperature (HT) partition function is given by

$$Z_{\text{HT}} = \sum_{\substack{\text{site} \\ \text{configurations}}} \prod_{\text{faces}} \exp(-\beta E_{jkl}). \quad (3)$$

We construct an array  $V$  by

$$V_{jkl} = \exp(-\beta E_{jkl}) = \begin{cases} 1 & j+k+l = 0 \pmod q \\ u & \text{otherwise} \end{cases} \tag{4}$$

and relate  $V$  to a ‘diagonalized’ form

$$V_{jkl} = \sum_m A_{jm} A_{km} A_{lm} \lambda_m \tag{5}$$

where

$$A_{jm} = q^{-1/3} \exp(ijm/q) \quad i = \sqrt{-1} \tag{6}$$

$$\lambda_0 = 1 + (q-1)u \tag{7}$$

$$\lambda_j = 1 - u \quad j \neq 0. \tag{8}$$

At each vertex there is a product

$$\prod_{j=1}^6 (A_{km_j} \lambda_{m_j})$$

to be summed over all values of  $k$ . The product gives

$$\begin{aligned} q^{-1} \prod_j \lambda_{m_j} & \quad \sum_j m_j = 0 \pmod q \\ 0 & \quad \text{otherwise.} \end{aligned}$$

This means that we can set up a graphical representation with triangles indexed by  $j$  having a contribution  $\lambda_j$ .

The high-temperature expansion is

$$Z_{HT} = \prod_{\text{sites}} q^{-1} \prod_{\text{faces}} [1 + (q-1)u] \sum_{\substack{\text{allowed} \\ \text{graphs}}} \prod_{\substack{\text{nonzero} \\ \text{faces}}} (1-u)/[1 + (q-1)u]. \tag{9}$$

Since the number of faces is twice the number of sites, the transformation

$$u \rightarrow (1-u)/[1 + (q-1)u]$$

transforms the low-temperature expansion for the partition function into the high-temperature expansions multiplied by  $\{q/[1 + (q-1)u]^2\}^{\text{number of sites}}$

If there is a unique transition point then it must occur at

$$u = (1-u)/[1 + (q-1)u],$$

whence

$$u_0 = (1 + \sqrt{q})^{-1}. \tag{10}$$

At this point the initial multiple  $q/[1 + u(q-1)]^2$  reduces to 1 as it must for equality of the partition function expressions. The initial factor  $q^2$  that appears in the low-temperature expression (2) becomes insignificant in the thermodynamic limit as will any of the neglected boundary corrections.

Since these expressions are equivalent to those for the  $q$ -state Potts model on the square lattice we can use the results of Potts (1952) and Kihara *et al* (1954) to give the critical energy  $E_c = 1 - q^{-1/2}$  assuming a continuous transition and taking the fully aligned state as the zero of energy. These results for  $u_c, E_c$  agree with previous work on the  $q = 2$  case for the triplet Ising model, but for other  $q$  values nothing is known of the behaviour of the model.

### 3. The triplet Ashkin–Teller analogue

For a four-state system it is possible to find a more general model which has a unique self-dual point. This model includes the four-state triplet model of § 2, and also a system of two independent triplet models as special cases. There is an obvious analogy with the Ashkin–Teller model (Ashkin and Teller 1943) which has as special cases the four-state Potts model and a system of two independent Ising models.

The vertex constraint is the same as the  $q = 4$  constraint used in § 2. The difference is that elements of types 1 and 3 are given weights different from those of type 2. The energy is

$$E_{jkl} = \begin{cases} 0 & j+k+l = 0 \pmod 4 \\ J & j+k+l = \pm 1 \pmod 4 \\ K & j+k+l = 2 \pmod 4. \end{cases} \tag{11}$$

The special cases are  $J = K$  corresponding to equation (1) and  $J = 2K$  corresponding to two independent triplet Ising models.

The same transformation  $A_{jm}$  is used to give the diagonal elements

$$\lambda_0 = 1 + 2u + w \tag{12a}$$

$$\lambda_1 = \lambda_3 = 1 - w \tag{12b}$$

$$\lambda_2 = 1 - 2u + w \tag{12c}$$

$$u = \exp(-\beta J)$$

$$w = \exp(-\beta K).$$

The low-temperature expansion is

$$Z_{LT} = q^2 \sum_{\substack{\text{allowed} \\ \text{graphs}}} \prod_{\substack{\text{faces} \\ 1,3}} u \prod_{\substack{\text{faces} \\ 2}} w \tag{13}$$

while the high-temperature expansion is

$$Z_{HT} = \prod_{\text{faces}} (\frac{1}{2}\lambda_0) \sum_{\substack{\text{allowed} \\ \text{graphs}}} \prod_{1,3} (\lambda_1/\lambda_0) \prod_2 (\lambda_2/\lambda_0). \tag{14}$$

It turns out that there is a self-dual point at which

$$\frac{1}{2}\lambda_0 = 1$$

$$\lambda_1/\lambda_0 = u$$

$$\lambda_2/\lambda_0 = w$$

and this point is given by

$$2u + w = 1. \tag{15}$$

For any positive  $J, K$  there will be a temperature at which (15) is satisfied.

For the special cases equation (15) will give the critical point if it is unique. In other cases the transformation will transform the low-temperature expressions into high-temperature expressions for a different pair of  $J, K$  values and the uniqueness assumption is not sufficient to show that (15) gives the critical point.

What can be done is to consider the point  $2J = K$  and then treat  $J_2 = \frac{1}{2}J - \frac{1}{4}K$  as a perturbation. Equation (15) gives

$$\frac{1}{4}K \partial\beta_c/\partial J_2 = -\beta_c\sqrt{2}. \tag{16}$$

The same value of  $\partial\beta_c/\partial J_2$  can also be obtained from perturbation expansions of the type considered by Kadanoff and Wegner (1971), and using the critical amplitudes given by Baxter and Wu (1974).

The perturbation expansion was performed by interpreting the energy (11) in terms of a triangular lattice with two spins  $\sigma_i, S_i = \pm 1$  at each site. Equation (11) corresponds to a Hamiltonian

$$\mathcal{H} = - \sum_{\text{triangles}} \frac{1}{4}K(\sigma_j\sigma_k\sigma_l + S_jS_kS_l) + J_2\sigma_j\sigma_k\sigma_l S_jS_kS_l. \tag{17}$$

While equation (16) suggests that equation (15) gives the critical point for sufficiently small  $J_2$ , for  $J_2 > \frac{1}{4}K$ , we can apply the symmetry arguments of Wegner (1972) to predict that two transitions will occur, as in the Ashkin–Teller model.

#### 4. The non-local random cluster expansion

In § 2 we defined the  $q$ -state triplet model in terms of a  $q$ -state local site variable and the vertex constraint in terms of a sum over indices that can take one of  $q$  values. This approach is obviously restricted to integral  $q$ . It has, however, been possible to generalize the  $q$ -state Potts model (Fortuin and Kasteleyn 1972) into a form with a two-state edge variable and no local constraint. The value of  $q$  appears as an arbitrary real parameter, which for integral  $q$  leads to a partition function equivalent to that of the  $q$ -state Potts model.

The random cluster model on an arbitrary lattice is defined by

$$Z = \sum_{C \subseteq E} (1-u)^{|C|} u^{|E-C|} q^{\gamma(C)}. \tag{18}$$

The set  $E$  is the set of all lattice edges so that the sum is over all weak subgraphs.  $\gamma(C)$  is the number of components in the graph  $C$ . This appears to be a completely free summation, but in fact for the square lattice we can use a global constraint to generate a transformation. We put

$$u = (1-w)/[1+(q-1)w] \tag{19}$$

to give

$$\begin{aligned} Z &= [1+(q-1)w]^{-|E|} \sum_{C \subseteq E} (qw)^{|C|} (1-w)^{|E-C|} q^{\gamma(C)} \\ &= \sqrt{q}/[1+(q-1)w]^{|E|} \sum_{C \subseteq E} w^{|C|} (1-w)^{|E-C|} q^{\gamma(C)+|C|-N} \end{aligned} \tag{20}$$

where  $N$  is the number of vertices,  $\frac{1}{2}|E|$ .

The constraint that enables us to complete the transformation is the global constraint given by the Euler formula

$$\text{Faces} = \text{components} + \text{edges} - \text{vertices} + 1 \tag{21}$$

or

$$f = \gamma(C) + |C| - N + 1. \tag{22}$$

If we transform the edges of a weak graph  $C$  into the absence of an edge on the dual lattice, and the edges of set  $E - C$  into a graph on the dual lattice, then the number of components of the dual lattice graph will be the number of faces of the original graph  $C$ .

In other words

$$Z(u) = q^{-1} \{ \sqrt{q/[1+(q-1)w]} \}^{|E|} Z(w). \tag{23}$$

Again, the initial factor becomes negligible and will in fact depend on the boundary conditions chosen. This duality transformation in a form appropriate to arbitrary  $q$  has been given previously by Baxter (1973).

The 'random cluster' formulation of the  $q$ -state triplet model is given by

$$Z = \sum_{C \leq F} (1-u)^{|C|} u^{|F-C|} q^{\delta(C)}. \tag{24}$$

The set  $C$  is a set of triangles and  $F$  is the set of all triangular faces.  $\delta(C)$  is obtained by considering a system whose only interactions are on the set of faces  $C$ .

Taking the number of ground states of such a system to be  $q^{\delta(C)}$  defines  $\delta(C)$ . We have not been able to find a way of expressing  $\delta(C)$  in terms of graph variables, and so we have no indication of how to go about setting up a duality transformation for non-integral  $q$ .

To show that equation (24) is in fact the partition function for the  $q$ -state triplet Potts model, we derive a recursion relation from equation (2) considering the partition function on a graph  $G$ . Splitting the summation over all configurations into terms with face  $e$  in the '0' ground state and terms with face  $e$  nonzero gives

$$q^{-2}Z(G) = \sum_G \prod_G u = \sum_{G:e=0} \prod_G u + u \sum_{G:e \neq 0} \prod_{G-e} u \tag{25}$$

and considering a graph  $G - e$ ,

$$q^{-2}Z(G - e) = \sum_{G-e:e=0} \prod_{G-e} u + \sum_{G-e:e \neq 0} \prod_{G-e} u \tag{26}$$

so that

$$Z(G) = q^2(1-u) \sum_{G:e=0} \prod_{G-e} u + q^2uZ(G - e). \tag{27}$$

The summations denote summing over all configurations and the products are over all perturbed faces of the graph  $G$  (or  $G - e$ ). Iteration of equation (27) leads to equation (24) with  $\delta(C)$  as described above.

### Conclusions

We have discussed some new classes of self-dual models with  $q$ -state variables and three-site interactions. It should be noted that on the face-centred cubic lattice an Ising system with pure four-spin interactions is self-dual (Wood 1972). This model can be generalized to a  $q$ -state system with the energy being 0 or  $J$  when, at each vertex,  $\sum t_i \bmod q$  is 0 or nonzero. The transformation  $A_{jm}$  is  $q^{-1/4} \exp(ijm/q)$ . The factor  $q^{-1}$  combines with the  $\lambda_0$  factors to give 1 at the self-dual point because the number of tetrahedra is twice the number of sites. Since series analysis for the  $q = 2$  case has been interpreted as indicating two transitions (Wood and Griffiths 1973) we do not pursue this model further. A random cluster form can be obtained, again with  $\delta(C)$  in terms of the ground states of  $C$ . This

type of analysis is complicated by the fact that the number of ground states of the system as a whole is of order  $q^{3\sqrt{N}}$ .

Apart from the interest in the self-dual models, the random cluster formulation of these models exhibits a number of interesting properties. This formulation appears to have suppressed all the symmetry of the  $q$ -state models so that the definition can be extended to non-integral  $q$ . In such a case it becomes inappropriate to ask how the critical behaviour depends on the symmetries of the model and some alternative approach is needed. The other interesting feature of the random cluster formulation is the fact that a single global constraint was used to generate the square lattice duality transformation. This approach is rather different to previously known transformations as formalized by Wegner (1972). It would be of considerable interest to obtain a similar transformation for the multi-site  $q$ -state models, but it has not been possible to find one as yet.

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